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Finitely conducting compressible aligned magnetofluiddynamic parallel flows

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Abstract. A method for studying compressible, aligned MFD parallel flows is discussed. Flow equations are recast in an orthogonal streamline coordinate system, by employing some results from differential geometry. Exact solutions of these equations are obtained by assuming, a priori, certain functional forms of the speed, for straight parallel flows. Several examples are given to illustrate this method.

1. Introduction

In this paper, we investigate the steady plane flow of a viscous compressible electrically conducting fluid of finite electrical conductivity in the presence of a magnetic field when the magnetic field and velocity field are everywhere parallel. Such a flow is governed by a system of non-linear partial differential equations. Various methods exist for solving non-linear partial differential equations. W.F. Ames [1] has given an excellent treatment of the various methods employed for solving these equations and their applications. We study, specifically, straight parallel flows and have adopted M.H. Martin's [2] approach to obtain exact integrals or solutions for these flows. Since the electrical conductivity of most fluids is finite, our accounting for finite electrical conductivity makes the flow problem realistic and attractive from both a mathematical and a physical point of view.

The exact integrals of the fully viscous and compressible flow equations are very rare. There exist two definitions of exact integrals. A set of functions for the unknowns is said to be an exact integral of the flow equations if these functions satisfy the equations. This is the first definition of an exact integral of flow equations. Even if these solutions do not provide flows with solid, fixed boundaries or with movable obstacles, they give us a glimpse of the nature of the solutions of the flow. An exact integral that does not seem to correspond to any real problem today may be found to correspond to one in the future. According to the second definition, a set of known functions satisfying the flow equations constitutes an exact integral if these functions give the solution to a problem with boundaries arising from a real, physical problem which could be realistically imposed. Accordingly, the domain occupied by the fluid will have to be bounded by fixed or movable solid on which proper conditions will have to be satisfied. We know of a limited number of exact integrals in fluid dynamics conforming to this definition. In this work, we have employed the first definition of exact integrals.

In incompressible viscous flow, there exists a unique flow field satisfying the flow equations for a given streamline pattern. In compressible flow, however, this is not the case. R.C. Prim [3] discovered the substitution principle for steady rotational flow of an inviscid, thermally non-conducting gas subject to no extraneous forces, establishing the existence of a multiplicity of flows having the same streamline pattern. According to this principle, for a gas having an equation of the form $\rho = P(p)S(s)$, a Prim gas, any flow field satisfying the flow equations is a member of an infinite group of flow fields sharing the same streamlines and pressure field, the velocity and density fields of the members of this group being related by an arbitrary scalar function which is constant along each individual streamline. P. Smith [4] extended this substitution principle to Magnetogasdynamics for the motion of an infinitely conducting gas subject to a magnetic field establishing the fact that for a Prim gas, there exists an infinite group of flow fields satisfying the flow equations and sharing the same streamlines and pressure field. In this work, for viscous compressible finitely conducting Magnetogasdynamic flows, we have obtained a multiplicity of flows for straight parallel streamlines. However, these flows do not seem to follow any substitution principle.

The plan of this paper is as follows: in section 2, the equations governing steady, plane, viscous, compressible Magnetogasdynamics flow are written in a suitable form. In section 3, we follow Martin [2] and employ some results from differential geometry to recast these equations when the streamlines and an arbitrary family of curves generate a coordinate net. In the final section, the governing equations for parallel straight flows are obtained when the streamline coordinate net is orthogonal. In this section, we derive various solution sets for our flow problem by investigating the following five different functional forms for the speed: $q = q(y)$, $q = q(x)$, $q = G(x) + F(y)$, $q = G(x)F(y)$ and $q = xG(y) + F(y)$.

2. Flow equations

The steady plane flow of a viscous compressible electrically and thermally conducting fluid of finite electrical conductivity, in the presence of a magnetic field, when the influence of radiation heat flux is negligible, is governed by the following system of seven equations [5]:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0\,, \tag{1}$$

$$\rho\Big(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\Big) + \frac{\partial p}{\partial x} = \frac{\nu}{3}\,\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \nu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) - \mu H_2\Big(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\Big)\,, \tag{2}$$

$$\rho\Big(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\Big) + \frac{\partial p}{\partial y} = \frac{\nu}{3}\,\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \nu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + \mu H_1\Big(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\Big)\,, \tag{3}$$

$$\begin{aligned}\nu\Big[2\Big(\frac{\partial u}{\partial x}\Big)^2 + 2\Big(\frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2 - \frac{2}{3}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)^2\Big] + K\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\Big) \\ = \rho\Big(u\,\frac{\partial e}{\partial x} + v\,\frac{\partial e}{\partial y}\Big) + p\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) - \frac{1}{\sigma}\Big(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\Big)^2\,,\end{aligned} \tag{4}$$

$$(uH_2 - vH_1) - \frac{1}{\mu\sigma}\Big(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\Big) = C\,, \tag{5}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0\,, \tag{6}$$

$$p = \rho R T\,. \tag{7}$$

Here the seven unknowns to be investigated are the two dynamic variables u, v, the two magnetodynamic variables H_1 and H_2, and the three thermodynamic variables p, ρ and T while ν, μ, K and σ are respectively the constant coefficient of viscosity, the constant magnetic permeability, the constant thermal conductivity and the constant electrical conductivity of the fluid.

The specific internal energy function e that enters into the energy equation is a known function of the thermodynamic variables for a given gas, namely, $e = e(T)$ for an ideal gas and $e = C_v T$ for an ideal polytropic gas where C_v is the specific heat at constant volume. Equation (5) is the diffusion equation, $\text{Curl}[(\mathbf{v} \times \mathbf{H}) - 1/\mu\sigma \,\text{Curl}\,\mathbf{H}] = 0$, for plane flow when C is the integration constant and equation (6) expresses the absence of magnetic poles in the flow. We investigate aligned flows so that the velocity field is everywhere parallel to the magnetic field in the flow plane. Using the definition of aligned flows, we have

$$\mathbf{H} = \rho f \mathbf{v}\,, \tag{8}$$

where $f(x, y)$ is some scalar field.

Employing (8) in equation (5), we get

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = -\mu\sigma C = j_0 \quad \text{(say)}\,. \tag{8$'$}$$

Introducing ω and P given by

$$\begin{aligned} \omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,, \\ P &= p - \frac{4}{3}\nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \end{aligned} \tag{9}$$

and employing (8) and (8′), the governing equations (1) to (7) may be replaced by

$$\begin{aligned} &\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0\,, \\ &\frac{1}{2}\rho\frac{\partial}{\partial x}(u^2+v^2) - \rho v\omega + \nu\frac{\partial \omega}{\partial y} + \mu j_0 f\rho v = -\frac{\partial P}{\partial x}\,, \\ &\frac{1}{2}\rho\frac{\partial}{\partial y}(u^2+v^2) + \rho u\omega - \nu\frac{\partial \omega}{\partial x} - \mu j_0 f\rho u = -\frac{\partial P}{\partial y}\,, \\ &\nu\left[\omega^2 + 4\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right] = \rho\left(u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y}\right) + P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \frac{j_0^2}{\sigma}\,, \\ &\rho f\omega + \rho\left(v\frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y}\right) + f\left(v\frac{\partial \rho}{\partial x} - u\frac{\partial \rho}{\partial y}\right) = j_0\,, \\ &u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0\,, \\ &\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,, \\ &P = \rho RT - \frac{4}{3}\nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \end{aligned} \tag{10}$$

The above system of equations (10) is a set of eight nonlinear partial differential equations in seven unknown functions u, v, f, ρ, T, ω and P of x, y. Having determined the unknown functions from equations (10) for a given problem, the pressure function is calculated by using the definition of P in (9). The advantage of this new system lies in the reduction of order from two to one in the linear momentum equations.

3. Alternate formulation of equations of motion

The equation of continuity in system (10) implies the existence of a streamfunction $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -\rho v , \qquad \frac{\partial \psi}{\partial y} = \rho u . \tag{11}$$

We take $\phi(x, y) = \text{constant}$ to be some arbitrary family of curves which generates with the streamlines $\psi(x, y) = \text{constant}$, a curvilinear net (ϕ, ψ) so that in the physical plane the independent variables x, y can be replaced by ϕ, ψ. In this section, we transform the equations of system (10) into a new form in the new independent variables ϕ, ψ. Let

$$x = x(\phi, \psi) , \qquad y = y(\phi, \psi) \tag{12}$$

define the curvilinear net with the squared element of arc length along any curve to be given by

$$\mathrm{d}s^2 = E(\phi, \psi)\,\mathrm{d}\phi^2 + 2F(\phi, \psi)\,\mathrm{d}\phi\,\mathrm{d}\psi + G(\phi, \psi)\,\mathrm{d}\psi^2 , \tag{13}$$

wherein

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 , \qquad F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi} \quad \text{and} \quad G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 . \tag{14}$$

Equation (12) can be solved to give ϕ, ψ as functions of x, y so that

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y} , \qquad \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y} , \qquad \frac{\partial y}{\partial \phi} = -J\frac{\partial \psi}{\partial x} \qquad \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x} , \tag{15}$$

where $0 < |J| < \infty$ and by (14)

$$J = \frac{\partial(x, y)}{\partial(\phi, \psi)} = \pm\sqrt{EG - F^2} = \pm W \quad \text{(say)}$$

is the transformation jacobian.

Denoting by α the local angle of inclination of the tangent to the coordinate line $\psi = \text{constant}$, directed in the sense of increasing ϕ, we have from differential geometry the following (cf. Martin [2]):

$$\frac{\partial x}{\partial \phi} = \sqrt{E}\cos\alpha , \qquad \frac{\partial y}{\partial \phi} = \sqrt{E}\sin\alpha , \tag{16}$$

$$\frac{\partial x}{\partial \psi} = \frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha , \qquad \frac{\partial y}{\partial \psi} = \frac{F}{\sqrt{E}} \sin \alpha + \frac{J}{\sqrt{E}} \cos \alpha , \tag{17}$$

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma^2_{11} , \qquad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E} \Gamma^2_{12} , \tag{18}$$

$$k = \frac{1}{W} \left[\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma^2_{11} \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma^2_{12} \right) \right] = 0 , \tag{19}$$

$$\frac{\partial}{\partial \phi} \left(\frac{E}{2W^2} \right) = \frac{1}{W^2} [F\Gamma^2_{11} - E\Gamma^2_{12}] , \tag{20}$$

$$\frac{\partial}{\partial \psi} \left(\frac{E}{2W^2} \right) = \frac{1}{W^2} [F\Gamma^2_{12} - E\Gamma^2_{22}] , \tag{21}$$

$$\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) = \frac{1}{W} [G\Gamma^2_{11} - 2F\Gamma^2_{12} + E\Gamma^2_{22}] , \tag{22}$$

where

$$\begin{aligned} \Gamma^2_{11} &= \frac{1}{2W^2} \left[-F \frac{\partial E}{\partial \phi} + 2E \frac{\partial F}{\partial \phi} - E \frac{\partial E}{\partial \psi} \right] , \\ \Gamma^2_{12} &= \frac{1}{2W^2} \left[E \frac{\partial G}{\partial \phi} - F \frac{\partial E}{\partial \psi} \right] , \\ \Gamma^2_{22} &= \frac{1}{2W^2} \left[E \frac{\partial G}{\partial \psi} - 2F \frac{\partial F}{\partial \psi} + F \frac{\partial G}{\partial \phi} \right] , \end{aligned} \tag{23}$$

and k is the Gaussian curvature. Having recorded the above results, we consider the equations in system (10).

The equation of continuity

Martin [2] has obtained the necessary and sufficient condition for the flow of a fluid along the coordinate lines $\psi = \text{constant}$ of a curvilinear coordinate system (12) with ds^2 given by (13), to satisfy the principle of conservation of mass as

$$\rho W q = \sqrt{E} , \qquad u + iv = \left(\frac{\sqrt{E}}{\rho J} \right) \exp(i\alpha) , \qquad q = \sqrt{u^2 + v^2} , \tag{24}$$

where $i = \sqrt{-1}$. Also, the fluid flows towards higher or lower parameter values of ϕ according as $J = \pm W$ is positive or negative. In the following work, we consider the fluid flowing towards higher parameter values of ϕ so that $J = W > 0$.

The linear momentum equations

On employing (11) in the linear momentum equations and taking $u^2 + v^2 = q^2$, we have

$$\frac{1}{2} \rho \left[\frac{\partial q^2}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial q^2}{\partial \psi} \frac{\partial \psi}{\partial x} \right] + \omega \frac{\partial \psi}{\partial x} + \nu \left[\frac{\partial \omega}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial \omega}{\partial \phi} \frac{\partial \psi}{\partial y} \right] - \mu j_0 f \frac{\partial \psi}{\partial x}$$
$$= - \frac{\partial P}{\partial \phi} \frac{\partial \phi}{\partial x} - \frac{\partial P}{\partial \psi} \frac{\partial \psi}{\partial x}$$

and

$$\frac{1}{2}\rho\left[\frac{\partial q^2}{\partial\phi}\frac{\partial\phi}{\partial y}+\frac{\partial q^2}{\partial\psi}\frac{\partial\psi}{\partial y}\right]+\omega\frac{\partial\psi}{\partial y}-\nu\left[\frac{\partial\omega}{\partial\phi}\frac{\partial\phi}{\partial x}+\frac{\partial\omega}{\partial\psi}\frac{\partial\psi}{\partial x}\right]-\mu j_0 f\frac{\partial\psi}{\partial y}$$
$$=-\frac{\partial P}{\partial\phi}\frac{\partial\phi}{\partial y}-\frac{\partial P}{\partial\psi}\frac{\partial\psi}{\partial y}.$$

Making use of the transformation equations (15) in the above equations, we get

$$\frac{1}{2}\rho\left[\frac{\partial q^2}{\partial\phi}\frac{\partial y}{\partial\psi}-\frac{\partial q^2}{\partial\psi}\frac{\partial y}{\partial\phi}\right]-\omega\frac{\partial y}{\partial\phi}+\nu\left[\frac{\partial\omega}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial\omega}{\partial\psi}\frac{\partial x}{\partial\phi}\right]+\mu j_0 f\frac{\partial y}{\partial\phi}$$
$$=-\frac{\partial P}{\partial\phi}\frac{\partial y}{\partial\psi}-\frac{\partial P}{\partial\psi}\frac{\partial y}{\partial\phi},$$

$$\frac{1}{2}\rho\left[-\frac{\partial q^2}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial q^2}{\partial\psi}\frac{\partial x}{\partial\phi}\right]+\omega\frac{\partial x}{\partial\phi}-\nu\left[\frac{\partial\omega}{\partial\phi}\frac{\partial y}{\partial\psi}-\frac{\partial\omega}{\partial\psi}\frac{\partial y}{\partial\phi}\right]-\mu j_0 f\frac{\partial x}{\partial\phi}$$
$$=\frac{\partial P}{\partial\phi}\frac{\partial x}{\partial\phi}-\frac{\partial P}{\partial\psi}\frac{\partial x}{\partial\phi}.$$

Multiplying these two equations by $\partial x/\partial\psi$, $\partial y/\partial\psi$, respectively and adding gives one equation; again, multiplying by $\partial x/\partial\phi$, $\partial y/\partial\phi$, respectively and adding gives the second equation of the following set of new but equivalent form of the linear momentum equations. The two new linear momentum equations, using (24) are

$$J\frac{\partial P}{\partial\phi}+\frac{\sqrt{E}}{2q}\frac{\partial}{\partial\phi}(q^2)+\nu\left[E\frac{\partial\omega}{\partial\psi}-F\frac{\partial\omega}{\partial\phi}\right]=0, \tag{25}$$

$$J\frac{\partial P}{\partial\psi}+\frac{\sqrt{E}}{2q}\frac{\partial}{\partial\psi}(q^2)+\nu\left[F\frac{\partial\omega}{\partial\psi}-G\frac{\partial\omega}{\partial\phi}\right]+J\omega-\mu j_0 fJ=0. \tag{26}$$

The vorticity equation

Chandna, Barron and Garg [6] have proved that the vorticity equation for the compressible fluids takes the form

$$\omega=\frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{\rho J}\right)-\frac{\partial}{\partial\psi}\left(\frac{E}{\rho J}\right)\right]=\frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{Fq}{\sqrt{E}}\right)-\frac{\partial}{\partial\psi}(\sqrt{E}q)\right]. \tag{27}$$

The current density equation

Using (17) in the equation for current density (8′), we get

$$\rho f\omega-\left[\frac{\partial\psi}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial f}{\partial y}\right]-\frac{f}{\rho}\left[\frac{\partial\psi}{\partial x}\frac{\partial\rho}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\rho}{\partial y}\right]=j_0.$$

Applying the transformation equation (15), the definitions (14) and the continuity equation (24), the current density in the new form in ϕ, ψ coordinates is given by

$$EJf\omega+q\sqrt{E}F\frac{\partial f}{\partial\phi}-qE\sqrt{E}\frac{\partial f}{\partial\psi}+FJfq^2\frac{\partial}{\partial\phi}\left(\frac{\sqrt{E}}{Jq}\right)-EJfq^2\frac{\partial}{\partial\psi}\left(\frac{\sqrt{E}}{Jq}\right)=q\sqrt{E}J^2 j_0. \tag{28}$$

The energy equation

On employing (11) in the energy equation, we get

$$\nu\left[\omega^2+4\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]=\left(\frac{\partial e}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial e}{\partial y}\frac{\partial \psi}{\partial x}\right)-\frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial \psi}{\partial x}\right)-K\nabla^2T-\frac{j_0^2}{\sigma}.$$

Using the transformation equations (15), equation (24) and the results (16) to (18), the energy equation in the new equivalent form in ϕ, ψ coordinates is

$$\nu J\left[\omega^2-\frac{4\Gamma_{12}^2}{E}q\frac{\partial q}{\partial \phi}+\frac{4\Gamma_{11}^2}{E}q\frac{\partial q}{\partial \psi}\right]=\frac{\partial e}{\partial \phi}-\frac{J^2q^2P}{E}\frac{\partial}{\partial \phi}\left(\frac{\sqrt{E}}{Jq}\right)-KJ\nabla^2T-\frac{Jj_0^2}{\sigma}, \tag{29}$$

where

$$\nabla^2T=\frac{1}{J}\left[\frac{\partial}{\partial \phi}\left\{\frac{G\frac{\partial T}{\partial \phi}-F\frac{\partial T}{\partial \psi}}{J}\right\}+\frac{\partial}{\partial \psi}\left\{\frac{E\frac{\partial T}{\partial \psi}-F\frac{\partial T}{\partial \phi}}{J}\right\}\right]. \tag{30}$$

The solenoidol and state equations

Using (11) in the solenoidal equation, we get

$$\frac{1}{\rho}\left[\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}+\frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x}\right]\frac{\partial \psi}{\partial y}-\frac{1}{\rho}\left[\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}+\frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial y}\right]\frac{\partial \psi}{\partial x}=\frac{1}{\rho}\frac{\partial(\phi,\psi)}{\partial(x,y)}\frac{\partial f}{\partial \phi}=0$$

which yields that

$$\frac{\partial f}{\partial \phi}=0 \quad \text{or} \quad f=f(\psi) \tag{31}$$

is the solenoidal condition in ϕ, ψ coordinates.

Making use of (11) in the state equation, we get

$$P=\rho RT+\frac{4\nu}{3\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial \psi}{\partial x}\right].$$

Applying transformation equations (15) and equation (24), we obtain

$$EJqP=E\sqrt{E}RT+\frac{4}{3}\nu q^3J^2\frac{\partial}{\partial \phi}\left(\frac{\sqrt{E}}{qJ}\right). \tag{32}$$

Summing up the results of this section, we have:

THEOREM I. *If the streamlines $\psi(x, y)$ = constant and an arbitrary family of curves $\phi(x, y)$ = constant generate a curvilinear net in the physical plane of a viscous compressible MFD fluid, then the flow in independent variables ϕ, ψ is governed by the following system:*

$$J\frac{\partial P}{\partial \phi}+\sqrt{E}\frac{\partial q}{\partial \phi}+\nu\left[E\frac{\partial \omega}{\partial \psi}-F\frac{\partial \omega}{\partial \phi}\right]=0, \tag{33}$$

$$J\frac{\partial P}{\partial \psi}+\sqrt{E}\frac{\partial q}{\partial \psi}+\nu\left[F\frac{\partial \omega}{\partial \psi}-G\frac{\partial \omega}{\partial \phi}\right]+J\omega-\mu Jfj_0=0\,, \tag{34}$$

$$\nu J\left[\omega^2-\frac{4\Gamma_{12}^2}{E}q\frac{\partial q}{\partial \phi}+\frac{4\Gamma_{11}^2}{E}q\frac{\partial q}{\partial \psi}\right]=\frac{\partial e}{\partial \phi}-\frac{J^2q^2P}{E}\frac{\partial}{\partial \phi}\left(\frac{\sqrt{E}}{Jq}\right)-KJ\nabla^2T-\frac{J^2j_0}{\sigma}\,, \tag{35}$$

$$EJf\omega+q\sqrt{E}F\frac{\partial f}{\partial \phi}-qE\sqrt{E}\frac{\partial f}{\partial \psi}+FJfq^2\frac{\partial}{\partial \phi}\left(\frac{\sqrt{E}}{Jq}\right)-EJfq^2\frac{\partial}{\partial \psi}\left(\frac{\sqrt{E}}{Jq}\right)=\sqrt{E}J^2qj_0\,, \tag{36}$$

$$\frac{\partial f}{\partial \phi}=0\,, \tag{37}$$

$$\omega=\frac{1}{J}\left[\frac{\partial}{\partial \phi}\left(\frac{Fq}{\sqrt{E}}\right)-\frac{\partial}{\partial \psi}(\sqrt{E}q)\right], \tag{38}$$

$$EJqP=E\sqrt{E}RT+\frac{4}{3}\nu q^3J^2\frac{\partial}{\partial \phi}\left(\frac{\sqrt{E}}{Jq}\right), \tag{39}$$

$$\frac{\partial}{\partial \psi}\left(\frac{J}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial \phi}\left(\frac{J}{E}\Gamma_{12}^2\right)=0 \tag{40}$$

of eight equation in eight unknowns E, F, G, ω, P, f, T and q as functions of ϕ, ψ.

Given a solution of this system, the density function, the flow in the physical plane and hodograph plane is described by (cf. Martin)

$$\rho=\frac{\sqrt{E}}{qJ}\,,\qquad z=\int\frac{e^{i\alpha}}{\sqrt{E}}\{Ed\phi+(F+iJ)\,d\psi\}\,, \tag{41}$$

$$\alpha=\int\frac{J}{E}(\Gamma_{11}^2\,d\phi+\Gamma_{12}^2\,d\psi)\,, \tag{42}$$

$$u+iv=\frac{\sqrt{E}}{\rho J}\exp(i\alpha)\,,\qquad H_1+iH_2=f(u+iv) \tag{43}$$

and

$$p=P+\frac{4}{3}\nu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]. \tag{44}$$

4. Parallel straight flows

To investigate straight parallel flows, we take the streamlines to be parallel to the x-axis and the (ϕ, ψ)-net to be an orthogonal net. By this choice and equation (13), we obtain $\phi=\phi(x)$, $\psi=\psi(y)$ along with

$$ds^2=dx^2+dy^2 \tag{45}$$

and

$$ds^2=E\phi'^2(x)\,dx^2+2F\phi'(x)\psi'(y)\,dx\,dy+G\psi'^2(y)\,dy^2\,, \tag{46}$$

where prime denotes the derivative with respect to the argument. Comparing equations (45) with (46), we have

$$E=\frac{1}{\phi'^2(x)}\,,\qquad F=0\,,\qquad G=\frac{1}{\psi'^2(y)}\,,\qquad J=\sqrt{EG-F^2}=\frac{1}{\phi'(x)\psi'(y)}\,. \tag{47}$$

Employing (47) in the first equation of (41), we get

$$\rho q=\psi'(y) \tag{48}$$

so that $q=q(y)$ only if the fluid is incompressible and this fact is elegantly employed in the study of viscous incompressible parallel flows by R. Berker [7] for first grade fluids and by Siddiqui and Kaloni [8] for third grade fluids. For fluids with compressibility, ρ is not a constant and, therefore, $q=q(x,y)$ in general.

Employing (47) in equations (33) to (40), we find that the Gauss equation is identically satisfied and we have

$$\nu\,\frac{\partial\omega}{\partial y}+\psi'(y)\,\frac{\partial q}{\partial x}=-\frac{\partial P}{\partial x}\,, \tag{49}$$

$$\nu\,\frac{\partial\omega}{\partial x}-\psi'(y)\left[\omega+\frac{\partial q}{\partial y}-\mu f j_0\right]=\frac{\partial P}{\partial y}\,, \tag{50}$$

$$C_v\psi'(y)\,\frac{\partial T}{\partial x}+P\,\frac{\partial q}{\partial x}-K\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]-\frac{j_0^2}{\sigma}=\nu\omega^2\,, \tag{51}$$

$$-\frac{\partial}{\partial y}\,[\psi'(y)f]=j_0\,, \tag{52}$$

$$\frac{\partial f}{\partial x}=0\,, \tag{53}$$

$$-\frac{\partial q}{\partial y}=\omega\,, \tag{54}$$

$$\frac{\psi'(y)RT}{q}-\frac{4}{3}\,\nu\,\frac{\partial q}{\partial x}=P\,. \tag{55}$$

We eliminate ω from equations (49) and (50) by using equation (54) and apply the integrability condition

$$\frac{\partial^2 P}{\partial x\,\partial y}=\frac{\partial^2 P}{\partial y\,\partial x}$$

to find that the speed $q(x,y)$ must satisfy

$$\nu\left[\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right]-\psi'(y)\,\frac{\partial q}{\partial x}=L(x)\,, \tag{56}$$

where $L(x)$ is an arbitrary function of x. Equation (56) is a linear elliptic partial differential equation in $q(x,y)$. However, this equation has two arbitrary functions $\psi'(y)$ and $L(x)$ as a coefficient and the nonhomogeneous term, respectively. Therefore, the approach of finding the general solution of (56) is not taken in the following analysis. Since by assuming, a priori, a functional form for the speed one may obtain a solution set for the flow problem, we derive various solution sets for our flow problem by investigating the following five different functional forms for the speed function:

1. $q = q(y)$: the speed is constant on each individual streamline.
2. $q = q(x)$: the flow is irrotational and the speed is constant on each orthogonal trajectory.
3. $q = G(x) + F(y)$: the addition form.
4. $q = G(x)F(y)$: the product form.
5. $q = xG(y) + F(y)$: the Riabouchinsky form in y.

EXAMPLE I. We take

$$q = q(y)\,; \qquad q'(y) \neq 0 \tag{57}$$

in equation (56) and obtain

$$q = C_1 y^2 + C_2 y + C_3\,, \qquad L(x) = 2\nu C_1\,, \tag{58}$$

where C_1, C_2 and C_3 are three arbitrary constants such that C_1 and C_2 are not simultaneously zero. Using (58) and (53) in equations (52) and (54) respectively, we get

$$\omega = -2C_1 y - C_2\,, \qquad f(y) = -\frac{1}{\psi'(y)}\,[C_4 + j_0 y]\,, \tag{59}$$

where C_4 is an arbitrary constant such that constants C_4 and j_0 cannot be simultaneously zero anywhere. Employing (58) and (59) in equations (48), (49), (50) and (55), we get the thermodynamic variables, after integrating for P, to be

$$\rho = \frac{\psi'(y)}{C_1 y^2 + C_2 y + C_3}\,, \tag{60}$$

$$P = p = 2\nu C_1 x - \frac{\mu}{2}\, j_0^2 y^2 - \mu j_0 C_4 y + C_5 \tag{61}$$

and

$$T = \frac{(C_1 y^2 + C_2 y + C_3)\Big(2\nu C_1 x - \frac{\mu}{2}\, j_0^2 y^2 - \mu j_0 C_4 y + C_5\Big)}{R\psi'(y)}\,, \tag{62}$$

where C_5 is an arbitrary constant.

Equations (58) to (62) and (8) define the solution set for parallel straight flows of finitely conducting fluids for the assumed functional form for the speed q given by (57) provided this solution set also satisfies the energy equation (51) and, therefore, we have

$$\frac{2C_1 K\nu}{R}\left[\frac{C_1 y^2 + C_2 y + C_3}{\psi'(y)}\right]'' x + \left[\frac{j_0^2}{\sigma} - \frac{K}{R}\left\{\left(\frac{C_1 y^2 + C_2 y + C_3}{\psi'(y)}\right)\left(\frac{\mu}{2}\, j_0^2 y^2 + \mu j_0 C_4 y - C_5\right)\right\}'' - \frac{2}{R}\,\nu C_1 C_v (C_1 y^2 + C_2 y + C_3) + \nu(4C_1^2 y^2 + 4C_1 C_2 y + C_2^2)\right] = 0\,.$$

Since this equation must hold true for all x, it follows that the coefficients of different powers of x must be zero. Hence, for our solution set defined by (58) to (62), we must have

$$\frac{2C_1 K\nu}{R}\left[\frac{C_1 y^2 + C_2 y + C_3}{\psi'(y)}\right]'' = 0 \tag{63}$$

and

$$\begin{aligned}&\frac{j_0^2}{\sigma} + \nu(4C_1^2 y^2 + 4C_1 C_2 y + C_2^2) - \frac{2\nu}{R} C_1 C_v (C_1 y^2 + C_2 y + C_3)\\&- \frac{K}{R}\left[\left(\frac{C_1 y^2 + C_2 y + C_3}{\psi'(y)}\right)\left(\frac{\mu}{2} j_0^2 y^2 + \mu j_0 C_4 y - C_5\right)\right]'' = 0 .\end{aligned} \tag{64}$$

Equations (63) and (64) hold true if either

$$C_1 = 0 , \qquad \frac{R}{K}\left(\nu C_2^2 + \frac{j_0^2}{\sigma}\right) = \left[\frac{C_2 y + C_3}{\psi'(y)}\left(\frac{\mu}{2} j_0^2 y^2 + \mu j_0 C_4 y - C_5\right)\right]'' \tag{65}$$

or

$$\psi'(y) = \frac{C_1 y^2 + C_2 y + C_3}{L_1 y + L_2} ,$$

$$\begin{aligned}&\frac{j_0^2}{\sigma} + \nu(4C_1^2 y^2 + 4C_1 C_2 y + C_2^2) - \frac{2\nu C_1 C_v}{R} (C_1 y^2 + C_2 y + C_3)\\&- \frac{K}{R}\left[\left(\frac{C_1 y^2 + C_2 y + C_3}{\psi'(y)}\right)\left(\frac{\mu}{2} j_0^2 y^2 + \mu j_0 C_4 y - C_4\right)\right]'' = 0 ,\end{aligned} \tag{66}$$

where L_1, L_2 are arbitrary constants.

From equations (65), we have

$$C_1 = 0 , \qquad \psi'(y) = \frac{2K(C_2 y + C_3)\left(\frac{\mu}{2} j_0^2 y^2 + \mu j_0 C_4 y - C_5\right)}{R\left(\nu C_2^2 + \frac{j_0^2}{\sigma}\right) y^2 + C_6 y + C_7} \tag{67}$$

so that solution set is given by (58) to (62) and (8) which C_1 and $\psi'(y)$ given by (67).

From equation (66), on eliminating $\psi'(y)$ between the two equations, we obtain

$$\begin{aligned}&\left(4\nu - \frac{2\nu}{R} C_v\right) C_1^2 y^2 + \left[4\nu C_1 C_2 - \frac{2\nu}{R} C_1 C_2 C_v - \frac{3\mu j_0^2 L_1 K}{R}\right] y\\&+ \left[\frac{j_0^2}{\sigma} + \nu C_2^2 - \frac{2\nu C_1 C_3 C_v + 2\nu j_0 C_4 L_1 K - \mu j_0^2 L_2 K}{R}\right] = 0 .\end{aligned}$$

Since this equation holds true for all values of y provided the coefficients of different powers of y are zero and since $C_v/R \neq 2$ for any polytropic gas and C_1 and C_2 must not be simultaneously zero, it follows that equations (58) to (62) and (8) define a solution set if

$$C_1 = L_1 = 0 , \qquad L_2 = \frac{-R\left(\frac{j_0^2}{\sigma} + \nu C_2^2\right)}{\mu j_0^2 K}$$

with $\psi'(y)$ given by the first equation of (66).

Summing up, if the speed has the functional form given by (57) for our finitely conducting flow, then we have

$$q = C_2 y + C_3\,, \qquad \rho = \frac{\psi'(y)}{C_2 y + C_3}\,,$$

$$H = -(C_4 + j_0 y)\,, \qquad p = C_5 - \mu j_0 C_4 y - \frac{\mu}{2} j_0^2 y^2\,,$$

$$T = \frac{(C_2 y + C_3)\left(C_5 - \mu j_0 C_4 y - \frac{\mu}{2} j_0^2 y^2\right)}{R\psi'(y)}\,,$$

where

$$\psi'(y) = \frac{C_2 y + C_3}{\frac{-R}{\mu j_0^2 K}\left(\frac{j_0^2}{\sigma} + \nu C_2^2\right)} \quad \text{or} \quad \psi'(y) = \frac{2K(C_2 y + C_3)\left(\frac{\mu}{2} j_0^2 y^2 + \mu j_0 C_4 y - c_5\right)}{R\left(\nu C_2^2 + \frac{j_0^2}{\sigma}\right) y^2 + C_6 y + C_7}\,.$$

Carrying out the mathematical analysis as in example I, we solve the flow problems corresponding to the other four forms. The results obtained for these four forms are given below as examples II to V.

EXAMPLE II. If $q = q(x)$ such that $q'(x) \neq 0$, then

either

$$q(x) = \frac{K(\gamma - 1)C_5}{C_1 \nu R \gamma} \exp\left[\left(\frac{\gamma}{\gamma - 1}\right)\left(\frac{C_1 \nu R}{K}\right)x\right] + C_6\,,$$

$$\rho = \frac{C_1^2 \nu R \gamma}{K(\gamma - 1)C_5 \exp\left[\left(\frac{\gamma}{\gamma - 1}\right)\left(\frac{C_1 \nu R}{K}\right)x\right] + C_1 C_6 \nu R \gamma}\,,$$

$$H = -(C_3 + j_0 y)\,,$$

$$p = \left[\frac{4}{3}\nu C_5 - \frac{C_5 K(\gamma - 1)}{R\gamma}\right]\exp\left[\frac{\gamma C_1 \nu R}{(\gamma - 1)K}x\right] - 2C_2 \nu \exp(C_1 x) - \mu j_0\left(C_3 y + \frac{j_0}{2} y^2\right) + (C_4 + C_1 C_6 \nu)\,,$$

$$T = \frac{p}{\rho R}\,,$$

where $\gamma = R/C_v + 1$ is the adiabatic constant, $C_1 \neq 0$, C_2, $C_3 \neq 0$, C_4, $C_5 \neq 0$, $C_6 = (C_1 \nu R)/(K\mu\sigma)$ and j_0 are arbitrary constants such that

$$C_1^4 = \frac{3K^4 \mu^2 j_0^2 (\gamma - 1)^2 \sigma}{\nu^4 R^3 \gamma [4\nu\gamma R - 3K(\gamma - 1)]} \quad \text{and} \quad K = \frac{8R\nu\gamma(1 - 2\gamma)}{3(\gamma - 1)(1 - 3\gamma)}\,;$$

or

$$q = q(x)\,,$$
$$\rho = \frac{C_1 \nu}{q(x)}\,,$$
$$H = -C_3\,,$$
$$p = C_4 + \frac{1}{3} C_2 \nu \exp(C_1 x) - C_1 \nu \int \exp(C_1 x)\left[\int \exp(-C_1 x)\{q''(x) - C_1 q'(x)\}\,\mathrm{d}x\right]\mathrm{d}x$$
$$+ \frac{4}{3} \nu \exp(C_1 x)\left[\int \exp(-C_1 x)\{q''(x) - C_1 q'(x)\}\,\mathrm{d}x\right],$$
$$T = \frac{p}{\rho R}\,,$$

where $C_1 \neq 0$, C_2, $C_3 \neq 0$ and C_4 are arbitrary constants and $q(x)$ is a solution of

$$q(x)q'''(x) - \left[\frac{C_1 \nu C_v}{K} q(x) - \frac{3C_4}{4\nu} + \frac{3C_1}{2} q(x) - 3q'(x)\right]q''(x)$$
$$- \left[\frac{3C_1 C_4 C_v}{4K} - \frac{3C_1^2 \nu C_v}{2K} q(x) + \frac{C_1 \nu C_v}{K} q'(x) + \frac{3C_1}{2} q'(x) + \frac{3C_1 C_4 R}{4K}\right.$$
$$\left. - \frac{3C_1^2 \nu R}{4K} q(x)\right]q'(x) = 0\,.$$

EXAMPLE III. If $q(x, y) = F(y) + G(x)$ such that $F'(y) \neq 0$ and $G'(x) \neq 0$ then $q(x, y) = C_2 y^2 + C_3 y + C_4 + G(x)$

$$\rho = \frac{C_1 \nu}{q(x, y)}\,,$$
$$H = -(j_0 y + C_5)\,,$$
$$p = 2\nu C_2 x - \nu C_1 G(x) - \mu j_0\left(\frac{j_0}{2} y^2 + C_5 y\right) + \frac{4}{3} \nu G'(x) + C_6\,,$$
$$T = \frac{p}{\rho R}\,,$$

where $G(x)$ is a solution of a second order linear equation with constant coefficient given by

$$\frac{K\mu j_0}{RC_1 \nu}\left[\frac{j_0 C_3}{2} - C_2 C_5\right]G''(x) + \left[\mu j_0 C_2 C_5\left(1 + \frac{C_v}{R}\right) - \frac{\mu j_0^2}{2} C_3\left(\frac{C_v}{R} + 1\right)\right]G'(x)$$
$$+ \frac{K\mu C_2 j_0}{RC_1 \nu}[6j_0 C_3 - 6C_2 C_5 - 3j_0 C_5] = 0\,,$$

when the solution must satisfy the following two equations:

$$-\frac{4KC_3}{3RC_1} G'''(x) + \left[\frac{4\nu C_v C_3}{3R} + \frac{K}{RC_1 \nu}(\mu j_0 C_5 + \nu C_1 C_3)\right]G''(x)$$
$$+ \left[-\frac{C_v}{R}(\mu j_0 C_5 + \nu C_1 C_3) + \mu j_0 C_5\right]G'(x) + \frac{2\nu C_v}{R} C_2 C_3$$
$$+ \frac{K}{RC_1 \nu}(6\mu j_0 C_2 C_5 + 3\mu j_0^2 C_5) - 4\nu C_2 C_3 = 0$$

and

$$\frac{C_v}{R}\Big[2\nu C_2 xG'(x) - \nu C_1 G(x)G'(x) + \frac{4}{3}\,\nu G'^2(x) + C_6 G'(x) + 2C_2C_4\nu - \nu C_1 C_4 G'(x)$$
$$+ \frac{4}{3}\,\nu C_4 G''(x) + 2\nu C_2 G(x) - \nu C_1 G(x)G'(x) + \frac{4}{3}\,\nu G(x)G''(x)\Big] + 2\nu C_2 xG'(x)$$
$$- \nu C_1 G(x)G'(x) + C_6 G'(x) - \frac{K}{RC_1\nu}\Big[2\nu C_2 xG''(x) - \nu C_1 G(x)G''(x) + \frac{4}{3}\,\nu G'(x)G''(x)$$
$$+ C_6 G''(x) + 4\nu C_2 G'(x) - 2\nu C_1 G'^2(x) + \frac{8}{3}\,\nu G'(x)G''(x) - \nu C_1 C_4 G''(x)$$
$$+ \frac{4}{3}\,\nu C_4 G'''(x) - \nu C_1 G(x)G''(x) + \frac{4}{3}\,G(x)G'''(x) + 4\nu C_2^2 x - 2\nu C_1 C_2 G(x)$$
$$+ \frac{8}{3}\,\nu C_2 G'(x) + 2C_2C_6 - 2\mu j_0 C_3 - \mu j_0^2 C_4 - \mu j_0^2 G(x)\Big] - \frac{j_0^2}{\sigma} - \nu C_3^2 = 0\,,$$

where $C_1 \neq 0$, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants.

EXAMPLE IV. If $q(x, y) = F(y)G(x)$ such that $F'(y) \neq 0$ and $G'(x) \neq 0$ then *either*

$$q(x, y) = C_2\, e^{C_1 x} F(y)\,,$$
$$\rho = \frac{\psi'(y)}{q(x, y)}\,,$$
$$H = -C_3\,,$$
$$p = C_2\, e^{C_1 x}\Big[\frac{\nu}{C_1}\,F''(y) - \psi'(y)F(y) + \frac{4}{3}\,\nu C_1 F(y)\Big] + C_4\,,$$
$$T = \frac{p}{\rho R}\,,$$

where $C_1 \neq 0$, $C_2 \neq 0$, C_3, C_4 are arbitrary constants and $F(y)$ and $\psi'(y)$ are the solutions of the following system of three non-linear ordinary differential equations:

$$\nu F''(y) + [\nu C_1^2 - C_1\psi'(y)]F(y) = C_5\,,$$
$$K\Big[\frac{F(y)}{\psi'(y)}\Big]'' + KC_1^2C_2\,\frac{F(y)}{\psi'(y)} - C_1(C_v + R)F(y) = 0\,,$$
$$K\Big[\frac{3C_5F + C_1^2F^2}{\psi'(y)}\Big]'' + 4C_1^2K\Big[\frac{3C_5F + C_1^2F^2}{\psi'(y)}\Big] + 3RC_1F'^2(y) + (3RC_1^3 - 2C_vC_1^3)F^2(y)$$
$$- (6C_vC_1C_5 + 3C_1C_5R)F(y) = 0\,.$$

Here C_5 is an arbitrary constant;

or

$$q(x, y) = F(y)G(x) ,$$

$$\rho = \frac{1}{F^2(y)G(x)} [\nu C_3 F(y) + C_4] ,$$

$$H = -(j_0 y + C_5) ,$$

$$p = \frac{1}{3} \nu F(y)G'(x) - C_4 G(x) + \nu C_2 \int G(x)\,\mathrm{d}x - \mu j_0\left(\frac{j_0}{2} y^2 + C_5 y\right) ,$$

$$T = \frac{p}{\rho R} ,$$

where $F(y)$ and $G(x)$ are solutions of

$$F''(y) - C_1 F(y) = C_2 ,$$

$$G''(x) - C_3 G'(x) + C_1 G(x) = 0 ,$$

and these unknown functions also must satisfy an equation in $F(y)$ and $G(x)$ obtained by eliminating the flow variables from the energy equation (51). Here C_1, C_2, C_3, C_4 and C_5 are arbitrary constants.

EXAMPLE V. If $q(x, y) = xG(y) + F(y)$ such that $G'(y) \neq 0$ and $F'(y) \neq 0$ then

$$q(x, y) = x(b_1 y + b_2) + F(y) ,$$

$$\rho = \frac{2Kb_3(b_1 y + b_2)}{[x(b_1 y + b_2) + F(y)][2Kb_3(b_6 y + b_7) - \nu R b_1^2 y^2]} ,$$

$$H = -(j_0 y + b_4) ,$$

$$p = \frac{1}{3} \nu(b_1 y + b_2) - \mu j_0\left(\frac{j_0}{2} y^2 + b_4 y\right) + b_3 x + b_5 ,$$

$$T = \frac{p}{\rho R} ,$$

where $b_1 \neq 0$, b_2, b_3, b_4, b_5, b_6 and b_7 are arbitrary constants and $F(y)$ satisfies the following three equations:

$$F''(y) = \frac{1}{\nu} b_3 + \frac{2Kb_3(b_1 y + b_2)^2}{2Kb_3 \nu(b_6 y + b_7) - \nu^2 R b_1^2 y^2} ,$$

$$Kb_3\left[\frac{F(y)}{\psi'(y)}\right]'' + 2\nu b_1 R F'(y) + \frac{2}{3} \nu K\left(b_1 b_6 - \frac{R\nu b_1^2 b_2}{2Kb_3}\right) - \frac{R\nu b_1^2 b_5}{b_3} - \mu j_0 K(j_0 b_7 + 2b_4 b_6)$$
$$- \mu j_0\left[-\frac{3R\nu b_1^2 j_0^2 y^2}{Kb_3} + 3\left(j_0 b_6 - \frac{R\nu b_1^2 b_4}{Kb_3}\right) y\right] - \frac{R\nu^2 b_1^3}{b_3} y - b_3(2C_v + R)(b_1 y + b_2) = 0 ,$$

$$\frac{K}{R}\left\{\left[\left(-\frac{\mu j_0^2}{2}\,y^2-\mu j_0 b_4 y+b_5\right)\frac{2Kb_3(b_6y+b_7)-R\nu b_1^2y^2}{2Kb_3(b_1y+b_2)}\,F(y)\right]''\right.$$
$$\left.+\frac{1}{3}\,\nu\left[\left(-\frac{R_\nu b_1^2}{2Kb_3}\,y^2+b_6y+b_7\right)F(y)\right]''\right\}-\nu F'^2(y)-\frac{C_v}{R}\,b_3F(y)$$
$$-\nu\left(\frac{C_v}{3R}-1\right)(b_1y+b_2)^2-\left(\frac{C_v}{R}+1\right)\left[-\mu j_0\left(\frac{j_0}{2}\,y^2+b_4y\right)+b_5\right](b_1y+b_2)$$
$$+\frac{2Kb_3}{R}\left(-\frac{R\nu b_1^2}{2Kb_3}\,y^2+b_6y+b_7\right)-\frac{j_0^2}{\sigma}=0\,.$$

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